6.6. Special Type of Matrices.

Def.-

A matrix A is positive definite if it is

(A) Symmetric and if (B) $\vec{x}^T A \vec{x} > 0$, for any $\vec{x} \in \mathbb{R}^n$.
s.t. $\vec{x} \neq \vec{0}$.

$$\vec{x}^T A \vec{x} = [x_1 \; x_2 \cdots x_n] \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= [x_1 \; x_2 \ldots x_n] \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{pmatrix} =$$

$$x_1 \left(\sum_{j=1}^{n} a_{1j} x_j \right) + x_2 \left(\sum_{j=1}^{n} a_{2j} x_j \right) + \cdots + x_n \left(\sum_{j=1}^{n} a_{nj} x_j \right)$$

$$= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

<u>Example:</u> $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ Is it symmetric?

$\vec{x} = (x_1, x_2, x_3)$

$$\Rightarrow \quad x^T A \vec{x} = [x_1 \; x_2 \; x_3] \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} =$$

$$= 2x_1^2 - \underbrace{x_1 x_2} - \underbrace{x_2 x_1} + 2x_2^2 - \underbrace{x_3 x_2} - \underbrace{x_2 x_3} + 2x_3^2 =$$

$$= 2x_1^2 - 2x_1 x_2 + 2x_2^2 - 2x_2 x_3 + 2x_3^2 =$$

$$= x_1^2 + (x_1^2 - 2x_1 x_2 + x_2^2) + (x_2^2 - 2x_2 x_3 + x_3^2) + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

for $\vec{x} \neq \vec{0}$.

Definition is hard to verify, in general.

Following theorem help to decide which matrices are not positive definite.

Thm.- $A_{n\times n}$ is positive definite, then

a) A is nonsingular.

b) $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$.

c) $\max\limits_{1\le k,j\le n} |a_{kj}| \le \max\limits_{1\le i\le n} |a_{ii}|$

d) $(a_{ij})^2 < a_{ii}\, a_{jj}$, $i \neq j$.

Proof.-

a) If A is Singular, then $A\vec{x} = \vec{0}$ has nontrivial solutions. There is $\underline{\vec{x}_0 \neq \vec{0}}$ such that

$\underline{A\vec{x}_0 = \vec{0}}$ $\Rightarrow$

$$\vec{x}_0^{\,T} A \vec{x}_0 = \vec{x}_0^{\,T}(A\vec{x}_0) = \vec{x}_0\ \vec{0} = \vec{0}$$

$\Rightarrow$ A is not $\underline{P.D}$.

b) Choose $\vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to x_i$ $\Rightarrow$ $[0 \ldots x_i\, 0 \ldots 0]\, A \begin{pmatrix} 0 \\ \vdots \\ x_i = 1 \\ \vdots \\ 0 \end{pmatrix} =$

$$= [0 \ldots \overset{=1}{x_i}\, 0 \ldots 0] \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ii} \\ \vdots \\ a_{ni} \end{pmatrix} = \overset{=1}{x_i}\, a_{ii} = a_{ii} \overset{\text{Hip.}}{>} 0.$$

Assume $A_{4\times 4}$

c) $k \neq j$, $\quad (k=3, \quad j=2)$

Define $\hat{x}$ as $\quad x_i = \begin{cases} 0, & i \neq j,\ i \neq k \\ 1, & i=j \\ -1, & i=k. \end{cases}$

In our case, $\quad \hat{x} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \begin{matrix} \\ \to i=2 \\ \to i=3 \\ \\ \end{matrix}$

$$\vec{x}^T A \vec{x} = [0 \ 1 \ -1 \ 0] \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= [0 \ 1 \ -1 \ 0] \begin{pmatrix} a_{12}-a_{13} \\ a_{22}-a_{23} \\ a_{32}-a_{33} \\ a_{42}-a_{43} \end{pmatrix} = (a_{22}-a_{23}) - (a_{32}-a_{33})$$

$$= a_{22}+a_{33} - (a_{23}+a_{32})$$

In general,

$$\vec{x}^T A \vec{x} = a_{jj} + a_{kk} - \underbrace{a_{jk} - a_{kj}}_{\shortparallel \ 2a_{kj}}$$

Symmetry + Pos. Def. } implies

$$0 < \vec{x}^T A \hat{x} = a_{jj} + a_{kk} - 2a_{kj}$$

$$\Rightarrow \boxed{2a_{kj} < a_{jj} + a_{kk}} \qquad (*)$$

Now define $\vec{z}$ such that

$$z_i = \begin{cases} 0, & i \neq j \text{ and } i \neq k \\ 1, & i = j \text{ or } i = k. \end{cases}$$

Then

$$0 < z^T A z = a_{jj} + a_{kk} + 2a_{kj}$$

$$\Rightarrow \quad \boxed{-2a_{kj} < a_{jj} + a_{kk}} \qquad (**)$$

$$\Rightarrow \quad 2a_{kj} > -(a_{jj} + a_{kk})$$

Combining $(*)$ and $(**)$.

$$-(a_{jj} + a_{kk}) < 2a_{kj} < a_{jj} + a_{kk}$$

$$\Rightarrow \qquad |a_{kj}| < \frac{a_{jj} + a_{kk}}{2} \leq \max_{1 \leq i \leq n} |a_{ii}|$$

$$\Rightarrow \quad \max |a_{kj}| \leq \max_{1 \leq i \leq n.} |a_{ii}|.$$

d) (in book).

Def. — A <u>leading principal submatrix</u> of a matrix A is a matrix of the form

$$A_K = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ \vdots & & & \\ a_{K1} & a_{K2} & \cdots & a_{KK} \end{pmatrix} \qquad 1 \le K \le n$$

Thm. — A symmetric matrix A is positive definite if and only if each of its leading principal submatrix has a positive determinant.

Example: Exercise (1) (f) $\quad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$

$$\det A_1 = \det [2] = 2 > 0$$

$$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = 7 > 0$$

$$\det A_3 = \det(A) = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 2 \end{vmatrix} = 16 - 2 - 2 - 8 = 4 > 0.$$

$\Rightarrow A$ is positive definite.

Thm.– A (Symm.) is positive definite if and only if Gaussian Elimination without row interchanges can be performed on the linear system $A\vec{x}=\vec{b}$, with all pivot elements positive.

Corollary 6.25 A Symm. is positive definite if and only if

$A = LDL^T$, where

L: Lower triang. with 1's on its diagonal.

D: diagonal matrix with positive diag. entries.

Corollary 6.26 A is positive definite $\Leftrightarrow$ $A = LL^T$

where

L: Lower Triang. with nonzero diag. entries.

Corollary 6.27 A Symm. for which G. Elim. can be performed without row interchanges. Then,

$$A = LDL^T$$

where

L: Lower triang with 1's on its diag.

D: Diag. matrix with $a_{11}^{(1)}\ a_{22}^{(2)} \dots a_{nn}^{(n)}$ on its diag.

Sect 6.6.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Exercise (2) (a) A is symmetric. and positive definite.

$A = LDL^T$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$LD = \begin{bmatrix} d_1 & 0 & 0 \\ l_{21} d_1 & d_2 & 0 \\ l_{31} d_1 & l_{32} d_2 & d_3 \end{bmatrix}$$

$$\Rightarrow \quad LDL^T = \begin{bmatrix} d_1 & 0 & 0 \\ l_{21} d_1 & d_2 & 0 \\ l_{31} d_1 & l_{32} d_2 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} = A$$

$$= \begin{bmatrix} d_1 & l_{21} d_1 & l_{31} d_1 \\ l_{21} d_1 & l_{21}^2 d_1 + d_2 & l_{21} d_1 l_{31} + d_2 l_{32} \\ l_{31} d_1 & l_{31} d_1 l_{21} + l_{32} d_2 & l_{31}^2 d_1 + l_{32}^2 d_2 + d_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Nonlinear Systems of 9 equations with 6 unknowns.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}$$

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$\boxed{d_1 = 2}$

$l_{21} d_1 = -1 \Rightarrow \boxed{l_{21} = -\frac{1}{2}}$

$l_{31} d_1 = 0 \Rightarrow \boxed{l_{31} = 0}$

$l_{21}^2 d_1 + d_2 = 2 \Rightarrow \frac{1}{4} 2 + d_2 = 2 \Rightarrow d_2 = 2 - \frac{1}{2} = \frac{3}{2}.$

$\boxed{d_2 = 3/2}$